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An Invariant Condition for Certain Automorphic Algebraic Forms.

BY A. B. COBLE.

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§1. *Introduction.*

Prof. Moore[*] has shown that every finite collineation group in n variables has for an invariant a certain Hermitian form which may by a suitable transformation of the variables be taken as

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \dots + x_n\bar{x}_n \tag{1}$$

Recalling that the collineations in contragredient variables are obtained by assuming the invariant

$$x_1u_1 + x_2u_2 + \dots + x_nu_n$$

and comparing this with (1) we see that when the group has been taken in Moore's canonical form, and the matrices of the collineation have a determinant value unity, then any matrix has for contragredient matrix its own conjugate. Or as Maschke[†] has pointed out, the minor of an element of the matrix is the conjugate of the element, and the complementary minor of a minor is the conjugate of the minor.

If the group has, for automorphic form of lowest degree, a certain unique algebraic form of order m in point coordinates, say f^m, the contragredient group has for automorphic form of lowest degree in contragredient variables a unique algebraic form of class m, say ϕ^m. If the group is in Moore's canonical form the coefficients of ϕ^m are the conjugates of those of f^m.

The collineation group being a thing of definite projective character, the same is true of the forms f^m and ϕ^m and it should be possible to define these forms by invariant conditions upon the general form of order m or of class m. This has actually been done for the three important groups in the linary domain and for the ternary G_{168}. But as f^m and ϕ^m occur simultaneously it would also seem possible to determine them simultaneously by invariant conditions upon a pair of general forms of order m and class m. In the following sucn an invariant condition will be given and will be seen to define some well-known automorphic forms.

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[*] Math. Ann. Bd. 50, p. 213.

[†] Math. Ann. Bd. 51, p. 253.

§2. *The Invariant Conditions.*

An algebraic form, f^{2n}, of even order $2n$ in q coordinates x may be written symbolically as $f^{2n}=(ax)^{2n}$; and a form γ^n of class n in contragredient variables as $\gamma^n=(u\gamma)^n$. γ^n defines then with regard to f^{2n} its "polar," c^n, where

$$c^n=(a\gamma)^n\,(ax)^n\equiv(cx)^n$$

Taking c^n and γ^n as contragredient "elements with the incidence condition, $(c\gamma)^n=0$" we see that f^{2n} defines a *correlation* in these elements. The correlation is better exhibited by writing f^{2n} in polarized form

$$f^{n,n}\equiv(ax)^n\,(ay)^n \tag{2}$$

Since $f^{n,n}$ is symmetric in x and y, the correlation is of the type known as the *polarity*. It is not, however, the most general polarity. The latter would be given by the symmetrical form

$$L^{n,n}=(lx)^n\,(my)^n=(ly)^n\,(mx)^n \tag{3}$$

which differs from $f^{n,n}$ in not being the polarized form of a $2n$-*ic*.

The inverse polarity of (3) is also a polarity

$$\Lambda^{n,n}=(u\lambda)^n\,(v\mu)^n=(u\mu)^n\,(v\lambda)^n \tag{4}$$

of the general type, *i. e.*, not the polar of a form of class $2n$. But this is also true in general of the polarity (2), *i. e.*, the correspondence of γ^n to c^n given by (2) is reversed by a form $\Lambda^{n,n}$ of the *general* type. Only when the form of f^{2n} satisfies certain conditions will the inverse polarity $\Lambda^{n,n}$ be the polarized form of a quantic of class $2n$. And there is suggested the obvious query:

What is the nature of the form f^{2n} when the correspondence it establishes between forms γ^n and c^n is reversed by a form, $\phi^{2n}=(u\alpha)^{2n}$, i. e., if $(a\gamma)^n\,(ax)^n=(cx)^n$ then $(c\alpha)^n\,(u\alpha)^n\equiv\lambda\,(u\gamma)^n$ or if

$$(ax)^n\,(a\alpha)^n\,(u\alpha)^n\equiv\lambda\,(ux)^n\,(a\alpha)^{2n}\,?^* \tag{5}$$

We assume of course that the correspondence is not degenerate, *i. e.*, that the determinant of the nth derivatives of f^{2n} is not zero. This determinant, which is called in certain connections the catalecticant of f, will be denoted by A. The

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[*] λ is here a numerical constant which by operation n times with the identical covariant (ux) we find to be $\dfrac{1}{n+q-1}$.

catalecticant bordered with two series of variables u and v in the combinations of class n gives $\Lambda^{n,n}$ and allowing the variables to coincide we obtain the form ϕ^{2n} as a contravariant of f^{2n}. From the symmetry of (5) we have that in the special case to be considered f^{2n} is the same contravariant of ϕ^{2n} that ϕ^{2n} is of f^{2n}. The forms f^{2n} and ϕ^{2n} will then be called a "covariant quantic-pair." To within a numerical factor A is the invariant $(a\alpha)^{2n}$.

The above query is answered completely in §3 for the binary domain and partially for the ternary domain in §4. Isolated results are further given and in §6 is found a treatment of a collineation group of order 25920 in five variables whose automorphic forms are obtained by the use of the invariant conditions just described.

§3. *The Forms f^{2n} and ϕ^{2n} in the Binary Domain.*

The distinction of contragredience is here trivial. We write both the forms f^{2n} and ϕ^{2n} [using as the symbol of a linear form $(a_1x_2 - a_2x_1)$] as

$$\begin{aligned} f^{2n} &\equiv (ax)^{2n} = (bx)^{2n} = \dots \\ \phi^{2n} &\equiv (\alpha x)^{2n} = (\beta x)^{2n} = \dots \qquad A \equiv (a\alpha)^{2n} \end{aligned} \tag{1}$$

and the condition (5) §2 becomes

$$(ax)^n \, (a\alpha)^n \, (\alpha y)^n \equiv \frac{1}{n+1} (a\alpha)^{2n} \, (xy)^n \tag{2}$$

From this we derive by the application of the Ω-process the further identities

$$\begin{aligned} &(a\alpha)^{n+1} (ax)^{n-1} (\alpha y)^{n-1} \equiv \frac{1}{n} (a\alpha)^{2n} (xy)^{n-1} \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &(a\alpha)^{n+l} (ax)^{n-l} (\alpha y)^{n-l} \equiv \frac{1}{n-l+1} (a\alpha)^{2n} (xy)^{n-l} \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &l = 0, 1, \dots n \end{aligned} \tag{3}$$

We consider now the product of an invariant, $(ab)^{2n}$, of f^{2n} by the same invariant $(\alpha\beta)^{2n}$ of ϕ^{2n}

$$\begin{aligned} (ab)^{2n} \, . \, (\alpha\beta)^{2n} &\equiv [(a\alpha)(b\beta) - (a\beta)(b\alpha)]^{2n} \\ &= \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (a\alpha)^{2n-k} (b\beta)^{2n-k} (a\beta)^k (b\alpha)^k \\ &+ \sum_{k=n}^{k=2n} (-1)^k \binom{2n}{k} (a\beta)^k (b\alpha)^k (a\alpha)^{2n-k} (b\beta)^{2n-k} \end{aligned}$$

In every term of the sum occurs a term with $(a\alpha)$ or $(a\beta)$ to a degree n or greater and applying the identities (3) we have

$$
\begin{aligned}
(ab)^{2n} \,.\, (\alpha\beta)^{2n} = & \sum_{k=0}^{k=n-1} (-1)^k \binom{2n}{k} \frac{1}{k+1} (a\alpha)^{2n} \,.\, (b\beta)^{2n} \\
& + \sum_{k=n}^{k=2n} (-1)^k \binom{2n}{k} \frac{1}{2n-k+1} (a\beta)^{2n} \,.\, (b\alpha)^{2n}
\end{aligned} \tag{4}
$$

All the symbols on the rights are symbols of A and we have

$$
(ab)^{2n} \,.\, (\alpha\beta)^{2n} \doteq \mu_n \,.\, A^2 \tag{5}
$$

where μ_n is a numerical value

$$
\mu_n = \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{1}{k+1} + \sum_{k=n}^{2n} (-1)^k \binom{2n}{k} \frac{1}{2n-k+1} \tag{6}
$$

We may obtain the value of the sum μ_n as follows:

$$
\begin{aligned}
\mu_n = & \; 2\left[1 - \tfrac{1}{2}\binom{2n}{1} + \tfrac{1}{3}\binom{2n}{2} \cdots\cdots + (-1)^{n-1} \frac{1}{n}\binom{2n}{n-1}\right] \\
& \qquad + (-1)^n \frac{1}{n+1}\binom{2a}{n} \\
(2n+1)\,\mu_n = & \; 2\left[\binom{2n+1}{1} - \binom{2n+1}{2} \cdots\cdots + (-1)^{n-1}\binom{2n+1}{n}\right. \\
& \qquad \left. + (-1)^n \binom{2n+1}{n+1}\right] - (-1)^n \binom{2n+1}{n+1} \\
2-(2n+1)\,\mu_n = & \; 2\left[1 - \binom{2n+1}{1} + \binom{2n+1}{2} \cdots\cdots + (-1)^{n+1}\binom{2n+1}{n+1}\right] \\
& \qquad + (-1)^n \frac{2n+1}{n+1} \\
= & \; 2\left[(-1^{n-1} \frac{(2n)\,!}{(n-1)\,!\;(n+1)\,!}\right] + (-1)^n \binom{2n+1}{n+1} \\
= & \; (-1)^{n-1} \frac{(2n)\,!}{(n-1)\,!\;(n+1)\,!}\left[2 - \frac{2n+1}{n}\right] \\
= & \; (-1)^n \frac{(2n)\,!}{n\,!\;(n+1)\,!}
\end{aligned}
$$

$$
\mu_n = \frac{2}{2n+1} + (-1)^{n+1} \frac{1}{(n+1)(2n+1)} \binom{2n}{n} \tag{7}
$$

We see then that when $n = 1$ or $n = 3$, $\mu_n = 1$; when $n = 2$, $\mu_n = 0$; when $n > 3$, μ_n is greater in absolute value than 1.

By a process exactly similar to that by which we obtained (5) we prove that

$$\begin{aligned} (\alpha\beta)^{2n} \,.\, (ax)^{2n} &= \mu_n \,.\, A \,.\, (\alpha x)^{2n} \qquad \text{and} \\ (ab)^{2n} \,.\, (\alpha x)^{2n} &= \mu_n \,.\, A \,.\, (ax)^{2n} \end{aligned} \tag{8}$$

Multiplying these two we see that

$$(\alpha\beta)^{2n} \,.\, (ab)^{2n} = \mu_n^2 \,.\, A^2$$

and comparing this with (5) we have at once

$$\mu_n = 1. \tag{9}$$

This argument is valid as long as $(ab)^{2n} \neq 0$ and $(\alpha\beta)^{2n} \neq 0$. But neither invariant vanishes unless $\mu_n = 0$ since A in (5) is by hypothesis not zero. The case $n = 2$ alone requires a special investigation. But it is well known that if a binary quartic $(ax)^4$ has its invariant $(ab)^4$ equal to zero, it has a covariant, the Hessian, which forms with it a "covariant quantic pair."* The quartic is in this case the well known "equianharmonic quartic" which is automorphic under a tetrahedral group.

We consider now the two cases for which $\mu_n = 1$. If $n = 1$, f^{2n} is a quadratic form and of course ϕ^{2n} coincides with it as we see also from (8). This case $n = 1$ occurs in every domain.

If $n = 3$, the covariant quantic pair is shown by (8) to be a single quantic which is subject to the condition (2) hence from (2) and (3)

$$\begin{aligned} (ax)^3\,(ab)^3\,(by)^3 &= \tfrac{1}{4}(ab)^6\,(xy)^3 \\ (ax)^2\,(ab)^4\,(by)^2 &= \tfrac{1}{3}(ab)^6\,(xy)^2 \end{aligned} \tag{10}$$

Putting $x = y$ in this last we have

$$(ab)^4\,(ax)^2\,(bx)^2 \equiv 0. \tag{11}$$

But (11) is the condition that the sextic $(ax)^6$ be that which is unaltered by an octahedral group.† And it is easily verified that the octahedral sextic satisfies the identity (2). Hence

(12) *In the binary domain there are only three types of covariant quantic-pairs; the quadratic taken with itself and automorphic under an infinite group; the equianharmonic quartic and its Hessian automorphic under the tetrahedral group; that sextic taken with itself which is automorphic under the octahedral group.*

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* Gordan, Invarianten Theorie, p. 211.

† Gordan, Invarianten Theorie, p. 211.

The case of the automorphic quartic presents nothing new. There are, however, some features in the case of the sextic that are novel. The general sextic $f^6 \equiv (ax)^6$ has for its catalecticant a symmetric determinant of the fourth order which, bordered with a series of variables x and a series y, gives rise to a symmetric form (12) $(dx)^3\,(ey)^3 \equiv (dy)^3\,(ex)^3$. In order that f^6 be the special form we are considering, this must be the polarized form of a sextic. The necessary and sufficient condition for this is given by the Clebsch–Gordan expansion as the identical vanishing of the quadratic form (13) $(de)^2\,(dx)\,(ex)$. Both these forms are of the third degree in the coefficients. Since the sextic has only one covariant quadratic of this degree we may say that

(14) *The necessary and sufficient conditions that a sextic be octahedral is the non-vanishing of the catalecticant A and the identical vanishing of the quadratic covariant of third degree in the coefficients.*

This criterion is superior to the one usually given (see (11)) in that the number of conditions shows the exact number of constants in the form.

Connected with the tetrahedral quartic is a set of three irrational covariant quadratics which play an important role. They are defined as the quadratics which are their own polars as to the quartic. The set may be generalized for the octahedral sextic by seeking for cubics whose polars as to the sextic are their own cubic covariants. We take the sextic in its canonical form

$$f^6 = x_1^5x_2 - x_2^5x_1$$

and the cubic as

$$P = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3\,.$$

The polar of P as to f^6 is the cubic covariant of P if

$$\begin{aligned} a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3 &= \lambda a_2 \\ 3(a_0a_1a_3 - 2a_0a_2^2 + a_1^2a_2) &= -\lambda a_3 \\ 3(a_0a_2a_3 - 2a_1^2a_3 + a_1a_2^2) &= \lambda a_0 \\ a_0a_3^2 - 3a_1a_2a_3 + 2a_2^3 &= -\lambda a_1. \end{aligned} \tag{15}$$

We may also add

$$a_0^2a_3^2 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2 - 6a_0a_1a_2a_3 = R \neq 0\,.$$

This system of equations * may be replaced by the system

$$\begin{array}{llll} 1^\circ & a_0a_1 + a_2a_3 = 0 & 4^\circ & a_0a_2^3 - a_1^3a_3 = 0 \\ 2^\circ & a_1^4 + a_2^4 = 0 & 5^\circ & a_0^2a_3^2 + 3a_1^2a_2^2 - 4a_0a_2^3 = 0 \\ 3^\circ & a_0^4 + a_3^4 = 0 & 6^\circ & (a_0a_3 - a_1a_2)^2 = \dfrac{R}{3}\,. \end{array} \tag{16}$$

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* The following solution was supplied by Mr. Landry.

We see at once that a_i cannot be zero unless P vanishes identically. Taking δ and ε so that $\delta^4 = -1$ and $\varepsilon^4 = -1$ we have from 2° and 3° that

$$a_2 = \varepsilon a_1 \qquad a_3 = \delta a_0.$$

Substituting these values in 1° or 4° we find

$$\delta\varepsilon = -1 \qquad \therefore \varepsilon = -\frac{1}{\delta} = \delta^3.$$

Substituting the values of a_2, a_3 and ε in 5° and putting $\dfrac{\delta a_1}{a_0} = \alpha$ the equation 5° becomes

$$3\alpha^4 + 4\alpha^3 + 1 = (\alpha + 1)^2 (3\alpha^2 - 2\alpha + 1) = 0$$

$\alpha = -1$ makes R vanish and must be discarded.

The other factor has roots $\alpha_1 = \dfrac{1 + i\sqrt{2}}{3}$ and $\alpha_2 = \dfrac{1 - i\sqrt{2}}{3}$. For the root α_2 and δ any one of the four primitive eighth roots of unity we find four cubics

$$\begin{aligned}
P_1 &= x_1^3 + 3\delta\alpha_2\, x_1^2x_2 + 3\delta^6\alpha_2\, x_1x_2^2 + \delta^7x_2^3\\
P_2 &= x_1^3 + 3\delta^3\alpha_2\, x_1^2x_2 + 3\delta^2\alpha_2\, x_1x_2^2 + \delta^5x_2^3\\
P_3 &= x_1^3 + 3\delta^5\alpha_2\, x_1^2x_2 + 3\delta^6\alpha_2\, x_1x_2^2 + \delta^3x_2^3\\
P_4 &= x_1^3 + 3\delta^7\alpha_2\, x_1^2x_2 + 3\delta^2\alpha_2\, x_1x_2^2 + \delta x_2^3
\end{aligned}$$

and for the root α_1 we find four others

$$\begin{aligned}
Q_1 &= x_1^3 + 3\delta^5\alpha_1\, x_1^2x_2 + 3\delta^6\alpha_1\, x_1x_2^2 + \delta^3x_2^3\\
Q_2 &= x_1^3 + 3\delta^7\alpha_1\, x_1^2x_2 + 3\delta^2\alpha_1\, x_1x_2^2 + \delta x_2^3\\
Q_3 &= x_1^3 + 3\delta\alpha_1\, x_1^2x_2 + 3\delta^6\alpha_1\, x_1x_2^2 + \delta^7x_2^3\\
Q_4 &= x_1^3 + 3\delta^3\alpha_1\, x_1^2x_2 + 3\delta^2\alpha_1\, x_1x_2^2 + \delta^5x_2^3.
\end{aligned}$$

Of this system we note that P_i and Q_i are cubic and cubic covariant. Each is the polar of the other as to f^6. But further the bilinear invariant of Q_i and P_k for $i \neq k$ is zero. This remarkable property is sufficient to define the system.

The two products $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ are each equal to

$$x_1^{12} - 33\, x_1^8x_2^4 - 33\, x_1^4x_2^8 + x_2^{12}$$

from which we see that when the sextic is taken as a regular octahedron* each set of four cubics represents the twelve middle points of the edges. These twelve points are arranged as follows: The octahedron has 4 planes of symmetry parallel to the 4 pairs of opposite faces. On each plane lie six mid-edge points.

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* Klein—Ikosaeder—I, 2.

Three of these that are equidistant say 1, 2, 3 may be chosen as P_1, the other three say 4, 5, 6 as Q_1. Each of the other planes contains a point of 1, 2, 3 and a point of 4, 5, 6. We choose as P_2, P_3, P_4 those sets of equidistant mid-edge points in each plane which do not contain 1, 2, or 3. The other sets of three are Q_2, Q_3, Q_4 respectively.

The condition (14) also admits of an interpretation in three dimensians when the cubic space curve C^3 is taken as the bearer of the binary value system. A binary cubic represents a point or a plane through the point. A general sextic represents a quadric Q which, if taken as a point quadric, must be apolar to the web of quadrics circumscribed to C^3. The condition (14) requires that Q also be apolar to the net of quadrics inscribed to Q. We have then

If Q is a quadric apolar to the ∞^2 quadrics containing the planes of C^3, the necessary and sufficient condition that Q cut out on C^3 an octahedral set of six points is that Q also be apolar to the ∞^2 quadrics containing the points of C^3. The relation of Q and C^3 is perfectly dual. The planes common to C^3 and Q osculate C^3 at the points common to Q and C^3.

There exists a single pair of "Möbius' tetrahedra" P_i and Q_i such that each is a self-polar tetrahedron of Q. The joins of corresponding points of P_i and Q_i are chords of C^3 meeting C^3 in the 8 points that correspond to the cube. The planes of P_i and of Q_i cut out on C^3 the twelve points that correspond to the mid-edge points of the cube or octahedron.

We obtain convenient canonical forms for our covariant quantic pairs by assuming f and ϕ both to be unaltered by the permutations and changes of sign of the variables and then applying the definition of the quantic pair to determine the remaining constants. The three cases are then the following

$$n = 1 \qquad f^2 = \phi^2 = x_1^2 + x_2^2$$

$$n = 2 \qquad f^4 = x_1^4 + x_2^4 \pm 2i\sqrt{3}\, x_1^2 x_2^2 \qquad \phi^4 = x_1^4 + x_2^4 \mp 2i\sqrt{3}\, x_1^2 x_2^2$$

$$n = 3 \qquad f^6 = \phi^6 = x_1^6 + x_2^6 - 5\,(x_1^4 x_2^2 + x_1^2 \times x_2^4)\,.$$

The latter is the octahedral sextic referred to a pair of opposite mid-edge points.

§4. *The Forms f^{2n} and ϕ^{2n} in the Ternary Domain.*

The simple arithmetical criterion [(9) §3] for a covariant quantic pair that was obtained for the binary domain does not admit of an immediate extension to the ternary domain. We shall treat the values of n separately and begin with n equal to 2.

In a former paper by the author* a certain class of biquadratic forms were treated which were defined in the same way as the above covariant form-pairs. The automorphic character of the resulting quartics was not recognized, however. We shall show first of all that the pair of quartics, f^4 and ϕ^4, are automorphic under the ternary group of collineations G_{168}. The method used is, in some respects, novel. The symbolic notation is that of Study's Ternären Formen.

The ternary quartic form, $f^4 = (ax)^4$, after being polarized into $(ax)^2(ay)^2$ may be viewed as a degenerate symmetric quadri-quadric form. As such it is distinguished from the general symmetric form $(ax)^2(by)^2 \equiv (ay)^2(bx)^2$ by the identical vanishing of its quadratic contravariant $(abu)^2$. If the ternary quadratic form be taken as fundamental space element, the biquadratic form appears as a quadric and under the general projective group, G_{35}, of this space has but three comitants which involve only one series of point or of dual coordinates. These are the quadric itself in point coordinates, the quadric in dual coordinates, and the discriminant of the quadric. In connection, then, with the given quartic $f^4 = (ax)^4$ we have the invariant

$$A \equiv |a_{ik,jl}| \qquad ij, \text{ and } kl = 11,\ 22,\ 33,\ 23,\ 31,\ 12 \tag{1}$$

and the form $(uA)^2(vB)^2 \equiv (uB)^2(vA)^2$ which is defined by the identity

$$(ax)^2(aA)^2(uB)^2 \equiv A\,.\,(ux)^2 \tag{2}$$

the three forms constituting the complete systen of $(ax)^2(ay)^2$ under G_{35}. They are of course also invariant under the projective group of the plane, G_8, a subgroup of G_{35}.

In A the element $a_{ij,kl}$ is distinguished by its position in the determinant from the element $a_{ik,jl}$. It seems useful in certain connections to preserve this distinction which will sometimes be emphasized by writing for the double suffices 11, 22, 33, 23, 31, 12 the single suffices 1, 2, 3, 4, 5, 6, respectively, in which case A appears as

$$|a_{i,k}| \qquad i, \text{ and } k = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

Thus the latter notation makes it possible to write $(uA)^2(vB)^2$ as the result of an evectant operator upon A namely

$$(uA)^2(vB)^2 \equiv (u\Omega^{(1)})^2(v\Omega^{(2)})^2 \tag{3}$$

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[*] Trans. Amer. Math. Soc. Vol. 4, No. 1, pp. 65-85; 1903. This paper will be referred to hereafter as "Trans."

where $\Omega^{(1)}_{ij}\ \Omega^{(2)}_{kl} = \mu \dfrac{\partial}{\partial a_{r,s}} A$, the numerical coefficient μ and the suffices r and s being determined by ij and kl. The symmetric form $(uA)^2(vB)^2$ determines the contravariant quartic form

$$\phi^4 \equiv (u\alpha)^4 \equiv (uA)^2(uB)^2 \tag{4}$$

and the covariant quadratic form

$$C \equiv (Cx)^2 \equiv (ABx)^2 \tag{5}$$

On the other hand, both (4) and (5) are required to determine (3), as is evident from the formula

$$(uB)^2(uB)^2 \equiv (u\alpha)^2(v\alpha)^2 + \tfrac{1}{3}(Cuv)^2 \tag{6}$$

It is also evident from (6) that the identical vanishing of C is necessary and sufficient in order that (3) be the polarized form of a quartic (4). Hence

(7) *The necessary and sufficient condition that the quartics f^4 and ϕ^4 be a covariant quantic pair is the identical vanishing of the conic C of the fifth degree in the coefficient of f^4.*

From (3) C may also be written as the result of a differential operation upon A

$$\begin{aligned}(Cx)^2 = \Big[&\Big(\frac{\partial}{\partial a_{23}} - \frac{1}{2}\frac{\partial}{\partial a_{44}}\Big)x_1^2 + \Big(\frac{\partial}{\partial a_{31}} - \frac{1}{2}\frac{\partial}{\partial a_{55}}\Big)x_2^2 + \Big(\frac{\partial}{\partial a_{12}} - \frac{1}{2}\frac{\partial}{\partial a_{66}}\Big)x_3^2 \\ &+ \Big(\frac{1}{2}\frac{\partial}{\partial a_{56}} - \frac{\partial}{\partial a_{14}}\Big)x_2x_3 + \Big(\frac{1}{2}\frac{\partial}{\partial a_{64}} - \frac{\partial}{\partial a_{25}}\Big)x_3x_1 \\ &+ \Big(\frac{1}{2}\frac{\partial}{\partial a_{45}} - \frac{\partial}{\partial a_{36}}\Big)x_1x_2\Big] A\end{aligned} \tag{8}$$

Writing this differential operator symbolically also we have

$$(Cx)^2 = (\omega x^2) \tag{9}$$

where ω_{ik} is then the symbol for an operation which is understood to be carried out on A.

We consider now the pencil of symmetric quadri-quadric forms

$$\lambda\,(ay)^2(az)^2 + \mu\,(\gamma yz)^2 \tag{10}$$

and form the invariant A of the pencil. By the use of the operator $(\omega x)^2$ defined in (8) this invariant takes the form

$$\begin{aligned}[\lambda^6 + (\omega\gamma)^2 \lambda^5 \mu + \frac{1}{2!} (\omega\gamma)^2 (\omega'\gamma')^2 \lambda^4 \mu^2 + \frac{1}{3!} (\omega\gamma)^2 (\omega'\gamma')^2 (\omega''\gamma'')^2 \gamma^3 \mu^3 + \\ \cdots\cdots + \frac{1}{5!} (\omega\gamma)^2 (\omega'\gamma')^2 \cdots\cdots (\omega^{\text{v}}\gamma^{\text{v}})^2 \mu^6] A\end{aligned} \tag{11}$$

This repetition of the operation ω is permissible since in the minors A, as in A itself, the distinction between such coefficients as $a_{44} = a_{23\,23}$ and $a_{23} = a_{22\,33}$ is preserved. The coefficient of each power of λ in (11) is a simultaneous invariant of $(ax)^4$ and $(u\gamma)^2$. Equating (9) to zero, we obtain in general six values of $\frac{\lambda}{\mu}$ to each of which corresponds a conic $(u\delta)^2$ such that

$$\lambda (a\delta)^2 (ay)^2 \equiv -\mu (\gamma \delta y)^2$$

i. e., the polar conic of $(u\delta)^2$ as to f^4 is its intermediate* with the fixed conic $(u\gamma)^2$. The particular case for which the conic $(u\gamma)^2$ is the square of a point x is of interest. There is then a web of conics $(u\delta)^2$—all pairs of points containing x—such that $(x\delta y)^2 \equiv 0$. Three values of λ in (11) are zero and the equation reduces to

$$\lambda^3 A + (\omega x)^2 \lambda^2 + \frac{1}{2} (\omega x)^2 (\omega' x)^2 \lambda \mu^2 + \frac{1}{6} (\omega x)^2 (\omega' x)^2 (\omega'' x)^2 \mu^3 = 0. \tag{12}$$

The coefficients of λ in (12) are important covariants of the quartic. Performing the operations indicated by the symbols ω we have

$$\begin{aligned}(\omega x)^2 = (Cx)^2 &= \left(2A_{\substack{2\\3}} - \frac{1}{2} A_{\substack{4\\4}}\right) x_1^2 + \cdots\cdot \\ (\omega x)^2 (\omega' x)^2 &= \left(A_{\substack{23\\32}} - A_{\substack{42\\43}}\right) x_1^4 + \cdots\cdot \\ (\omega x)^2 (\omega' x)^2 (\omega'' x)^2 &= -1/2\, A_{\substack{234\\324}} x_1^6 + \cdots\cdot\end{aligned} \tag{13}$$

the notation for the minors of A being that of Gordan† (Kerschensteiner). By means of these leading coefficients the covariants may be indentified with the following symbolic forms

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* "Trans." p. 70.

† Invarianten theorie, Erster Theil, p. 47.

$$
\begin{aligned}
C &= (\omega x)^2 &&= -\frac{1}{32}(abc)^2(ade)^2(dbc)(ebc)(dx)(ex) \\
S &= (\omega x)^2(\omega' x)^2 &&= -\frac{1}{12}(abc)(abd)(acd)(bcd)(ax)(bx)(cx)(dx) \qquad (14) \\
H &= (\omega x)^2(\omega' x)^2(\omega'' x)^2 &&= \frac{1}{2}(abc)^2(ax)^2(bx)^2(cx)^2
\end{aligned}
$$

We seek first to interpret the vanishing of the discriminant of the equation (12) *i. e.* of

$$2(3AS - 2C^2)(2HC - S^2) - (3AH - SC)^2 = 0 \qquad (15)$$

For a given x to each root of (12) corresponds a conic $(u\delta)^2$ whose polar as to f^4 is the two tangent of $(u\delta)^2$ through x. The point equation of $(u\delta)^2$ is the conic C formed for the pencil

$$\lambda^2(\omega y)^2 + \lambda\mu(\omega x)^2(\omega' y)^2 + \frac{1}{2}\mu^2(\omega x)^2(\omega' x)^2(\omega'' y)^2 + \psi(x, y) = 0 \qquad (16)$$

where $\psi(x, y)$ vanishes identically for $x = y$. Hence, if (16) passes through x, the first three terms separately vanish for $x = y$. But these terms are the derivative of 12 as to μ, *i. e.*, (12) has a double root and (15) is satisfied by x. Suppose now u is a line of the quartic ϕ^4 touching ϕ^4 at x. Then* u touches its associate* conic $(u\delta)^2$ at the point x and for a proper λ and μ $(u\delta)^2$ satisfies the identity

$$\lambda(a\delta)^2(ay)^2 \equiv \rho(uy)^2 \equiv -\mu(x\delta y)^2$$

and also passes through x. Hence x is a point of (15) and we have

(17) *The equation* (15) *is that of the contravariant quartic* $\phi^4 = (u\alpha)^4$ *in point coordinates.* We can easily show further that

(18) *The three pairs of lines through a general point x such that each pair touches its associate conic form the sextic covariant of the binary quartic formed by the four tangents from x to ϕ^4.*

If x is taken at a cusp of $(u\alpha)^4$, the four tangents include the cusp tangent three times, the sextic covariant the cusp tangent six times and (11) has a triple root. Hence

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* "Trans." p. 67, footnote.

(19) *The cusps of* ϕ^4 *are the intersections of* $3AS - 2C^2 = 0$ *and* $3(AH - SC) = 0$.

Returning to the general invariant (9) we can express in terms of its symbols other comitants of f^4; for example

$$\begin{array}{llr} (\omega\omega' u)^2 \equiv C_{402} & (\omega\omega' u)^2 (\omega x)^2 \equiv C_{322} & \\ (\omega\omega'\omega'')^3 \equiv C_{300} & (\omega\omega' u)^2 (\omega''\omega''' u)^2 \equiv C_{204} & (20) \\ (\omega\omega' u)^2 (\omega'' x)^2 (\omega''' x)^2 \equiv C_{242} & (\omega\omega'\omega'')^2 (\omega''' x)^2 (\omega^{\text{iv}} x)^2 \equiv C_{140} & \end{array}$$

C^{ikl} denoting a comitant of f^4 of degree i, order k, and class l. All of the comitants enumerated in (14) and (20) are expressible in terms of A, and the coefficients of $\phi^4 = (u\alpha)^4$ and of $(Cx)^2$ as follows:—

If λ and μ are values which satisfy (11) a conic $(u\delta)^2$ exists which satisfies the identity

$$(\gamma\delta y)^2 \equiv -\frac{\lambda_1}{\mu_1} (a\delta)^2 (ay)^2$$

Operating with both sides of this identity on $(uA)^2 (vB)^2$ and making use of (2) there results a new identity

$$(A\gamma\delta)^2 (uB)^2 \equiv -\frac{\lambda_1}{\mu_1} \cdot A \cdot (u\delta)^2$$

$-\frac{\lambda_1}{\mu_1} A$ is then a root of the characteristic equation of the connex (2, 2)

$$(A\gamma x)^2 (uB)^2$$

which can be written as *

$$\rho^6 - I_1 \rho^5 + I_2 \rho^4 - I_3 \rho^3 + I_4 \rho^2 - I_5 \rho + I_6 = 0 \text{ if}$$

$$I_k \equiv \frac{1}{k!} \begin{vmatrix} (AB\gamma)^2 & (AB'\gamma)^2 & (AB''\gamma)^2 & \dots \\ (A'B\gamma)^2 & (A'B'\gamma)^2 & (A'B''\gamma)^2 & \dots \\ (A''B\gamma)^2 & (A''B'\gamma)^2 & (A''B''\gamma)^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \qquad (21)$$

the determinant having k rows and columns. Putting $\rho = -\frac{\lambda_1}{\mu_1} A$ in the characteristic equation and dividing out by A^5 it becomes

$$A\lambda_1^6 + I_1 \lambda_1^5 \mu_1 + \frac{I_2}{A} \lambda_1^4 \mu_1^2 + \frac{I_3}{A^2} \lambda_1^3 \mu_1^3 + \dots + \frac{I_6}{A^5} \mu_1^6 = 0$$

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* "Trans." p. 73-4.

λ_1, and μ_1, satisfy (22) due to their satisfying (11) whence (11) and (22) are the same equation. Equating coefficients and introducing the values of I_k from (21) we have the relations

$$\begin{aligned}(\omega\gamma)^2 &= (AB\gamma)^2\\ A\,.\,(\omega\gamma)^2\,(\omega'\gamma')^2 &= [(AB\gamma)^2]^2-(AB'\gamma)^2\,(A'B\gamma')^2\\ A^2\,.\,(\omega\gamma)^2\,(\omega'\gamma')^2\,(\omega''\gamma'')^2 &= [(AA\gamma)^2]^3-3\,(AB\gamma)^2\,.\,(AB'\gamma)^2\,(A'B\gamma')^2\\ &\quad+2\,(AB'\gamma)^2\,(A'B''\gamma')^2\,(A''B\gamma'')^2.\end{aligned} \tag{23}$$

. . . .

The second of these identities will be treated first. It is of the second degree in the coefficients of $(u\gamma)^2$ but may be polarized by the introduction of a new set of coefficients of $(u\delta)^2$. The symbols γ and δ may then be replaced by x and y and we have

$$A\,.\,(\omega x)^2\,(\omega'y)^2\equiv(Cx)^2\,.\,(Cy)^2-(AB'x)^2\,(A'By)^2. \tag{24}$$

From the identity (6) we have

$$(uA)^2\,(vB)^2\,.\,(u'A')^2\,(v'B')^2\equiv\left[(u\alpha)^2\,(v\alpha)^2+\frac{1}{3}\,(Cuv)^2\right]$$
$$[(u'\alpha')^2\,(v'\alpha')_2+1/3\,(C'u'v')^2.$$

Adjoining on both sides the symbols u and v' into symbols x and then the symbols v and u' into symbols y there results on the left $(AB'x)^2\,(A'By)^2$ and on the right

$$(\alpha\alpha'x)^2\,(\alpha\alpha'y)^2+2/3\,(\alpha xy)^2\,(C\alpha)^2+\frac{1}{9}\,[-2\,(C'C'xy+3\,(C'x)^2\,.\,(C'y)^2]. \tag{25}$$

Substituting this value in (24) we have the final identity

$$\begin{aligned}A\,.\,(\omega x)^2\,(\omega y)^2\equiv-(\alpha\alpha'x)^2\,(\alpha\alpha'y)^2-2/3\,(\alpha xy)\,(C\alpha)^2+\frac{1}{9}\,[2\,(CC'xy)^2\\ +6\,(Cx)^2\,(C'y)^2].\end{aligned} \tag{26}$$

This gives at once the two special identities

$$\begin{aligned}A\,.\,(\omega x)^2\,(\omega'x)^2 &= \frac{2}{3}\,[(Cx)^2]^2-(\alpha\alpha'x)^4\\ A\,.\,(\omega\omega'u)^2 &= 4/3\,(CC'u)^2-2\,(u\alpha)^2\,(C'\alpha)^2.\end{aligned} \tag{27}$$

A similar process may be carried through for the third identity. The values obtained are rather cumbersome and we give only the following special results

$$A^2\,.\,(\omega x)^2\,(\omega'y)^2\,(\omega''z)^2\equiv2\,(\alpha\alpha'x)^2\,(\alpha'\alpha''y)^2\,\alpha''\alpha z)^2+\chi\,(C) \tag{28}$$

where $X(C)$ indicates a sum of comitants of at least the first degree in the coefficients of $(Cx)^2$

$$A^2 \,.\, (\omega x)^2 (\omega' x)^2 (\omega'' x)^2 \equiv 2 (\alpha\alpha' x)^2 (\alpha'\alpha'' x)^2 (\alpha''\alpha x)_2 \qquad\qquad (29)$$
$$- (Cx)^2 \,.\, (\alpha\alpha' x)^4 + 2/9 \, [(Cx)^2]^3.$$

From (29) and (27) the point equation of $(u\alpha)^4$ can be calculated from the form $[(\alpha\alpha' x)^4]^3 - 6[(\alpha\alpha' x)^2 (\alpha'\alpha'' x)^2 (\alpha''\alpha x)^2]^2$ and is found to coincide with that of (15).

Adjoining in (28) the variables of y and z to the variable u we have

$$A^2 \,.\, (\omega x)^2 (\omega'\omega'' u)^2 \equiv 2(\alpha\alpha'\alpha'') (\alpha\alpha' x)^2 (u\alpha'')^2 + \chi_1(C). \qquad (30)$$

A simple calculation of the value of $(\omega x)^2 (\omega'\omega'' u)^2$ from the original definition of the symbols ω gives

$$(\omega x)^2 (\omega'\omega'' u)^2 = -\frac{1}{48} [(aa'a'')^4 (ux)^4 + 6(aa'a'')^2 (aa'u)^2 (a''x)^2]. \qquad (31)$$

We suppose now $(Cx)^2$ identically zero.

The second formula of (27) shows that the contravariant conic of fourth degree, C_{402}, vanishes. This necessitates the vanishing of the entire pencil of covariant conics of fifth degree for C itself and the polar of C_{402} as to $(ax)^4$ are two distinct members of the pencil. *Hence, if $(Cx)^2 \equiv 0$, all the quadratic comitants of f^4 of fourth and fifth degrees also vanish identically.*

In (30), $\chi_1(C)$ vanishes and from (30) and (31) we have

$$-\frac{A^2}{48} [(aa'a'')^4 (ux)^2 + 6(aa'a)^2 (aa'u)^2 ({}''x)^2] = 2(\alpha\alpha'\alpha'') (\alpha\alpha' x)^2 (u\alpha'')^2. \qquad (32)$$

Taking the bilinear invariant as to u and x on both sides we obtain

$$- A^2 \,.\, (aa'a'')^4 = 8(\alpha\alpha'\alpha''). \qquad (33)$$

We recall also the theorem * that if the I_1 of $(u\alpha)^4$ is the I_2 of $(ax)^4$, the I_2 of $(u\alpha)^4$ is $\left(\frac{A^4}{64}\right)^n$ times the I_1 of $(ax)^4$ where n is the degree of I_1 in the coefficients. Treating the right hand member of (32) as I and remembering that the A of $(u\alpha)^4$ is now $\frac{A^5}{64}$ the theorem gives rise to the new identity

$$-\frac{1}{48}\left(\frac{A^5}{64}\right)^2 [(\alpha\alpha'\alpha'')^4 (ux)^2 + 6(\alpha\alpha'\alpha'')^2 (\alpha\alpha' x)^2 u a^2]$$
$$= \left(\frac{A^4}{64}\right)^3 2(aa'a'')^2 (aa'u)^2 (a''x)^2. \qquad (34)$$

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* "Trans." p. 68,

Replacing in this $(\alpha\alpha'\alpha'')^4$ by its value in (33) we find

$$-\frac{A^2}{48}\left[-2(aa'a'')^4\,(ux)^2+24(aa'a'')\,(a'a''u)\,(ax)^2\right]=2(\alpha\alpha'\alpha'')^2\,(\alpha'\alpha''x)^2\,(u\alpha)^2. \quad (35)$$

Comparing (35) and (32) we have finally

$$6(aa'a'')^2\,(a'a''u)^2\,(ax)^2\equiv(aa'a'')^4\,(ux)^2. \quad (36)$$

If then $C\equiv 0$ the identity (36) is true. Conversely, if (36) is true, $C\equiv 0$. For the quartic defined by the identity (36) whose complete system has been set forth by Gordan * has no quadratic comitants. The condition (36) may then be replaced by the condition, $C\equiv 0$. (36) is further shown by Gordan (loc. cit.) to be the necessary and sufficient condition that f^4 be automorphic under the finite ternary collineation group G_{168}. We have then

(37) *The vanishing of the covariant conic C of fifth degree is the necessary and sufficient condition that the quartics f^4 and ϕ^4 be automorphic under the group G_{168}.*

We obtain a convenient canonical form of the covariant quantic-pair by assuming quantics unaltered by a cyclical advance of the variables and by changes of sign of the variables. Taking f^4 as

$$a(x_1^4+x_2^4+x_3^4)+6b(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2)$$

and ϕ^4 as

$$\alpha(u_1^4+u_2^4+u_3^4)+6\beta(u_2^2u_3^2+u_3^2u_1^2+u_1^2u_2^2)$$

we find as the conditions that the polar of any conic as to f^4 shall have for polar as to ϕ^4 the given conic

$$a\alpha+2b\beta=4b\beta=\lambda\neq 0$$
$$a\beta+b\alpha+b\beta=0.$$

Putting $a=\alpha=1$ we find b and β as the two roots of

$$2\mu^2+\mu+1=0. \quad (38)$$

Putting then

$$\mu_1=\frac{-1+i\sqrt{7}}{4}\quad \mu_2=\frac{-1-i\sqrt{7}}{4}$$

we have for the covariant quantic-pair

$$\left.\begin{aligned} f^4&\equiv x_1^4+x_2^4+x_3^4+6\mu_1\,(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2)\\ \phi^4&\equiv u_1^4+u_2^4+u_3^4+6\mu_2\,(x_2^2u_3^2+u_3^2u_1^2+u_1^2u_2^2).\end{aligned}\right\} \quad (39)$$

This canonical form of the quartic unaltered by G_{168} has been mentioned by

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* Math. Ann., Vol. 17, p. 217 and p. 359.

Wiman[*] and the group itself has been treated in great detail by Klein and Fricke.[†] It may be obtained, however, with so little effort from the forms (39) that it seems worth while to carry through the work here.

We will seek first a correlation K which will transform f^4 into ϕ^4. The well known properties of the octahedral group G_{24} which evidently leaves f^4 and ϕ^4 unaltered suggests a trial of the correlation

$$K \equiv \begin{cases} x_1 = u_1 + u_2 \\ x_2 = u_1 - u_2 \\ x_3 = \qquad\quad \rho u_3. \end{cases}$$

In order that K transform f^4 into ϕ^4, ρ must satisfy the equations

$$2 + 6\mu_1 = \rho^4, \quad 1 - \mu_1 = \mu_1\rho^2, \quad \mu_2 = \frac{2\mu_1}{\rho^2}$$

whence $\rho = \pm \dfrac{1}{\mu_2}$. Choosing the negative sign and forming the combination KTK^{-1} where T is the collineation $u_1 = u_3'$, $u_2 = u_2'$, $u_3 = -u_1'$, which leaves ϕ^4 unaltered, we have

$$KTK^{-1} \equiv x' = \begin{pmatrix} 1 & -1 & -2\mu_2 \\ -1 & 1 & -2\mu_2 \\ 2\mu_1 & 2\mu_1 & 0 \end{pmatrix} x'$$

a collineation which leaves f^4 unaltered. Introducing a factor $\frac{1}{2}$ to make the determinant unity, interchanging the last two rows of the matrix and changing the sign of the first we have a collineation S of period 7 which also leaves f unaltered S and its powers are

$$S \equiv x' = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & 2\mu_2 \\ 2\mu_1 & 2\mu_1 & 0 \\ -1 & 1 & -2\mu_2 \end{pmatrix} x \qquad S^2 \equiv x' = \tfrac{1}{2}\begin{pmatrix} -2\mu_2 & -1 & 1 \\ 2\mu_2 & -1 & 1 \\ 0 & 2\mu_1 & 2\mu_1 \end{pmatrix} x$$

$$S^3 \equiv x' = \tfrac{1}{2}\begin{pmatrix} 2\mu_2 & 1 & 1 \\ 0 & -2\mu_1 & 2\mu_1 \\ 2\mu_2 & -1 & -1 \end{pmatrix} x \qquad S^4 \equiv x' = \tfrac{1}{2}\begin{pmatrix} 2\mu_1 & 0 & 2\mu_1 \\ 1 & -2\mu_2 & -1 \\ 1 & 2\mu_2 & -1 \end{pmatrix} x$$

$$S^5 \equiv x' = \tfrac{1}{2}\begin{pmatrix} -2\mu_1 & 2\mu_1 & 0 \\ -1 & -1 & 2\mu_2 \\ 1 & 1 & 2\mu_2 \end{pmatrix} x \qquad S^6 \equiv x' = \tfrac{1}{2}\begin{pmatrix} -1 & 2\mu_2 & -1 \\ 1 & 2\mu_2 & 1 \\ 2\mu_1 & 0 & -2\mu_1 \end{pmatrix} x$$

$$S^7 = x' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$$

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[*] Math. Annalen, Vol. 47, 1896, p. 536.

[†] Klein-Fricke; Elliptischen Modulfunctionen.

The G_7 made up of the powers of S, combined with the G_{24} consisting of the permutations and changes of sign of the variables gives the entire G_{168} or more briefly $G_{168} = G_{24} \cdot G_7$.

Any one of the 168 matrices is obtained from one of the above 7 by interchanging the rows or changing the signs of a row throughout. The factor $\frac{1}{2}$ before the matrix is understood as affecting each element and the group is then in Moore's canonical form.

In the theory of the group as developed by Klein-Fricke two sets of seven conics play an important role. We obtain the one set by transforming $x_1^2 + x_2^2 + x_3^2$ by G_7, the other by transforming this set by K. We find them to be

$$\begin{gathered} x_1^2 + x_2^2 + x_3^2 \\ -(\mu_2 + 1)\,x_1^2 - \frac{\mu_1}{2}(x_2^2 + x_3^2) \pm (2 + \mu_1)\,x_2x_3 \\ -(\mu_2 + 1)\,x_2^2 - \frac{\mu_1}{2}(x_3^2 + x_1^2) \pm (2 + \mu_1)\,x_3x_1 \\ -(\mu_2 + 1)\,x_3^2 - \frac{\mu_1}{2}(x_1^2 + x_2^2) \pm (2 + \mu_1)\,x_1x_2. \end{gathered} \tag{40}$$

The other set of seven is

$$\begin{gathered} 2\mu_1^2\,u_1^2 + u_2^2 + u_3^2 \\ u_1^2 + 2\mu_1^2\,u_2^2 + u_3^2 \\ u_1^2 + u_2^2 + 2\mu_1^2\,u_3^2 \end{gathered} \tag{41}$$

$$\frac{\mu_1 - 1}{4}(u_1^2 + u_2^2 + u_3^2) - \frac{3\mu_1 - 1}{2}(\pm u_2u_3 \pm u_3u_1 \pm u_1u_2)$$

the product of the three ambiguous signs in the last conic being 1.

These sets of conics have remarkable invariant properties. For either set the following facts are true:

The contragredient form of each conic has coefficients conjugate to those of the conic. The discriminant of each one of a set of seven has the same value. The bilinear invariant of any two (either being taken in point and the other in line form) is the same constant. This property is sufficient to determine the system. The fourteen conics are their own polars as to f^4 or ϕ^4 and are the only conics with this property.*

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* That is, each of the conics in lines has for polar as to f^4 the same conics in points.

The Ternary Covariant Quantic-Pair of Sextics.

It seems quite impracticable to treat the sextic case by a method analogous to that used for the quartic owing to the little knowledge at hand of the comitants of the sextic and the complicated nature of the invariant condition which is, in this case, the identical vanishing of a form $(ax)^2(u\alpha)^2$ of the tenth degree in the coefficients of the sextic. But we may begin with a simplified pair of quantics and find the form they take when required to be a covariant quantic pair. Assuming then, as in all previous cases two sextics unaltered by a cyclical advance of the variables and by the changes of sign of the variables we put

$$\begin{aligned} f^6 = a\,(x_1^6 + x_2^6 + x_3^6) &+ 15b\,(x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2) \\ &+ 15c\,(x_1^4x_3^2 + x_2^4x_1^2 + x_3^4x_2^2) + 90d\,x_1^2x_2^2x_3^2 \\ \phi^6 = \alpha\,(u_1^6 + u_2^6 + u_3^6) &+ 15\beta\,(u_1^4u_2^2 + u_2^4u_3^2 + u_3^4u_1^2) \\ &+ 15\gamma\,(u_1^4u_3^2 + u_2^4u_1^2 + u_3^4u_2^2) + 90\delta\,u_1^2u_2^2u_3^2. \end{aligned}$$

The conditions that every cubic shall have a polar cubic as to f^6 whose polar cubic as to ϕ^6 is the original cubic take the form

$$\begin{array}{rll} a\alpha + 3b\beta + 3c\gamma = \lambda & a\beta + 3b\gamma + 3c\delta = 0 & \\ 9b\beta + 9d\delta + 3c\gamma = \lambda & a\gamma + 3b\delta + 3c\beta = 0 & c\beta + 3b\delta + 3d\gamma = 0 \\ 9c\gamma + 9d\delta + 3b\beta = \lambda & c\alpha + 3b\gamma + 3d\beta = 0 & b\gamma + 3c\delta + 3d\beta = 0 \\ 36d\delta = \lambda & b\alpha + 3c\beta + 3d\gamma = 0. & \end{array}$$

The first set of four reduce to

$$2a\alpha = 45d\delta \qquad 4b\beta = 4c\gamma = 9d\delta$$

whence none of the unknowns can be zero unless $\lambda = 0$, a possibility that must be excluded. The determinant of the last set of two in the unknowns d and δ is $b\beta - c\gamma = 0$, hence the two must be proportional, *i. e.*

$$b = \sigma c, \ \gamma = \sigma\beta, \ c\beta = \sigma b\beta \text{ whence } \sigma^3 = 1.$$

Replacing c and γ by their values in terms of b and β the original system reduces to the following

$$\begin{array}{ll} 2a\alpha = 45d\delta & \dfrac{a}{b} = -3\sigma\left(1 + \sigma\dfrac{\delta}{\beta}\right) \\ 4b\beta = 9d\delta & \dfrac{\alpha}{\beta} = -3\sigma^2\left(1 + \sigma^2\dfrac{d}{b}\right). \end{array}$$

Put $b = \beta = 1$ and we have again

$$9d\delta = 4,\ a\alpha = 10,\ a = -3\sigma(1 + \sigma\delta),\ a = -3\sigma^2(1 + \sigma^2 d).$$

If we put $a = \sigma a$, $\alpha = \sigma^2 a$, $\sigma\delta = \delta$, and $\sigma^2 d = d$ this system reduces to that obtained by taking σ as the real cube root of unity. For this last case we find easily the values of a, α, d, and δ and obtain this pair of quantics after dividing through by a and α

$$\begin{aligned} f^6 &= \Sigma x_1^6 - a_1 \Sigma x_1^4 (x_2^2 + x_3^2) + 4a_2 x_1^2 x_2^2 x_3^2 \\ \phi^6 &= \Sigma u_1^6 - a_2 \Sigma u_1^4 (u_2^2 + u_3^2) + 4a_1 u_1^2 u_2^2 u_3^2 \end{aligned} \tag{42}$$

where

$$a_1 = \frac{3(5 + i\sqrt{15})}{4} \quad a_2 = \frac{3(5 - i\sqrt{15}}{4}.$$

This is the pair of sextics (see Wiman, loc. cit.) unaltered by a ternary collineation group G_{360} and we have

(43) *A necessary condition that a pair of sextics, f^6 and ϕ^6, be unaltered by the ternary finite collineation group, G_{360} is that the sextics form a covariant quantic-pair.*

It is also extremely probable that this condition is sufficient.

The other two solutions of the above equations which arise when σ is an imaginary cube root of unity are obtained from the given solution (42) by the collineation $x_1^1 = x_1$, $x_2^1 = \sigma x_2$, $x_3^1 = \sigma^2 x_3$ and are not essentially distinct from (42).

We obtain the G_{360} in the same way as G_{168}, *i. e.*, by determining ρ in the correlation K

$$K \equiv \begin{cases} x_1 = u_1 + u_2 \\ x_2 = u_1 - u_2 \\ x_3 = \qquad\ \rho u_3 \end{cases}$$

so that K will transform f^6 into ϕ^6. The conditions are

$$\frac{2 - 2a_1}{1} = \frac{\rho 6}{1} = \frac{30 + 2a_1}{-a_2} = \frac{-2a_1 \rho^4}{-a_2} = \frac{(-2a_1 + 4a_2)\rho^2}{-a_2} = \frac{(-12a_1 - 8a_2)\rho^2}{4a_1}$$

and the value $\pm \rho = \dfrac{\sqrt{5} + i\sqrt{3}}{2}$ satisfies all. We choose the positive sign, putting

$$\rho = \frac{\sqrt{5} + i\sqrt{3}}{2} \quad \bar{\rho} = \frac{\sqrt{5} - i\sqrt{3}}{2}$$

and transform by K the collineation, T,

$$u_1' = u_3 \qquad u_2' = u_2 \qquad u_3' = -u_1$$

which leaves ϕ^6 unaltered.

$$KTK' \equiv x' \equiv \tfrac{1}{2}\begin{pmatrix} 1 & -1 & \bar{\rho} \\ -1 & 1 & \bar{\rho} \\ -\rho & -\rho & 0 \end{pmatrix} x$$

a collineation which leaves f^6 unaltered. Interchanging the last two rows of the matrix and changing the sign of the second column we have the collineation

$$S_a \equiv x' = \tfrac{1}{2}\begin{pmatrix} 1 & 1 & \bar{\rho} \\ -\rho & \rho & 0 \\ -1 & -1 & \bar{\rho} \end{pmatrix} x$$

The lines whose coordinates are the rows of the matrix of S_a form a perspective triangle of f^6. The triangle is unaltered by changing the sign of x_3 and also by interchanging x_2 and x_3. We obtain from it by using G_{24} only five other triangles and the corresponding matrices are

$$S_b \equiv x' = \tfrac{1}{2}\begin{pmatrix} \bar{\rho} & 1 & 1 \\ 0 & -\rho & \rho \\ \bar{\rho} & -1 & -1 \end{pmatrix} x, \quad S_c \equiv x' = \tfrac{1}{2}\begin{pmatrix} 1 & \bar{\rho} & 1 \\ \rho & 0 & -\rho \\ -1 & \bar{\rho} & -1 \end{pmatrix} x,$$

$$S_d \equiv x' = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & -\bar{\rho} \\ \rho & \rho & 0 \\ 1 & -1 & -\bar{\rho} \end{pmatrix} x, \quad S_e \equiv x' = \tfrac{1}{2}\begin{pmatrix} -\bar{\rho} & -1 & 1 \\ 0 & \rho & \rho \\ -\bar{\rho} & 1 & -1 \end{pmatrix} x,$$

$$S_f \equiv x' = \tfrac{1}{2}\begin{pmatrix} 1 & -\bar{\rho} & -1 \\ \rho & 0 & \rho \\ -1 & -\bar{\rho} & 1 \end{pmatrix} x$$

S_a is a collineation of period 5. Introducing the notation

$$\varepsilon = \frac{-1 + i\sqrt{3}}{2}, \quad \varepsilon^2 = \frac{-1 - i\sqrt{3}}{2}, \quad a = \frac{-1 + \sqrt{5}}{2} \quad \bar{a} = \frac{-1 - \sqrt{5}}{2}$$

the square of S_a can be written

$$S_1 \equiv x' \equiv \tfrac{1}{2}\begin{pmatrix} -a & -\varepsilon^2 & \varepsilon\bar{a} \\ -\varepsilon^2\bar{a} & \varepsilon a & -1 \\ \varepsilon & \bar{a} & \varepsilon^2 a \end{pmatrix} x.$$

The triangle of the matrix S_1 is unaltered by the collineation $x_1 = x_2'$, $x_2 = -x_3'$, $x_3 = x_1'$. G_{24} will transform it into only 7 other triangles and we have for the corresponding matrices

$$S_2 \equiv x' = 1/2\begin{pmatrix} \varepsilon\bar{a} & -a & -\varepsilon^2 \\ -1 & -\varepsilon^2\bar{a} & \varepsilon a \\ \varepsilon^2 a & \varepsilon & \bar{a} \end{pmatrix}x, \quad S_3 \equiv x' = 1/2\begin{pmatrix} -\varepsilon^2 & \varepsilon\bar{a} & -a \\ \varepsilon a & -1 & -\varepsilon^2\bar{a} \\ \bar{a} & \varepsilon^2 a & \varepsilon \end{pmatrix}x,$$

$$S_4 \equiv x' = 1/2\begin{pmatrix} -a & \varepsilon^2 & -\varepsilon\bar{a} \\ -\varepsilon^2\bar{a} & -\varepsilon a & 1 \\ \varepsilon & -\bar{a} & -\varepsilon^2 a \end{pmatrix}x \quad S_5 \equiv x' = 1/2\begin{pmatrix} a & -\varepsilon\bar{a} & \varepsilon^2 \\ \varepsilon^2\bar{a} & 1 & -\varepsilon a \\ -\varepsilon & -\varepsilon^2 a & -\bar{a} \end{pmatrix}x,$$

$$S_6 \equiv x' = 1/2\begin{pmatrix} -\varepsilon\bar{a} & \varepsilon^2 & a \\ 1 & -\varepsilon a & \varepsilon^2\bar{a} \\ -\varepsilon^2 a & -\bar{a} & -\varepsilon \end{pmatrix}x, \quad S_7 \equiv x' = 1/2\begin{pmatrix} \varepsilon^2 & a & -\varepsilon\bar{a} \\ -\varepsilon a & \varepsilon^2\bar{a} & 1 \\ -\bar{a} & -\varepsilon & -\varepsilon^2 a \end{pmatrix}x,$$

$$S_8 \equiv x' = 1/2\begin{pmatrix} a & \varepsilon\bar{a} & -\varepsilon^2 \\ \varepsilon^2\bar{a} & -1 & \varepsilon a \\ -\varepsilon & \varepsilon^2 a & \bar{a} \end{pmatrix}x. \quad \text{We add also} \quad S_0 \equiv x' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}x.$$

S_2 and S_3 arise from S_1 by advancing its columns cyclically; S_4 arises from S_1 by changing the sign of the last two columns; and finally $S_5, \ldots . S_8$ from $S_1, \ldots . S_4$ respectively by interchanging the last two columns and changing all the signs.

The G_{360} is obtained by multiplying the G_{24} by the fifteen collineations S.

The collineations S are all in Moore's canonical form, *i. e.*, they have determinants unity and each element is the conjugate of its minor. The same is true of the entire group if G_{24} be taken in the canonical form.

The conics unaltered by the various octahedral and ikosahedral groups in the G_{360} are easily written. We have first a set of six conics "in involution," *i. e.*, such that each is a polar to any other in the two possible ways. These conics each unaltered by an ikosahedral group are of the form

$$I \qquad \begin{gathered} y_1^2 + \varepsilon y_2^2 + \varepsilon^2 y_3^2 \\ \frac{\rho^2 - 1}{4}(y_1^2 + y_2^2 + y_3^2) - \frac{\rho^2 + 1}{4} 2\,(y_2 y_3 + y_3 y_1 + y_1 y_2) \end{gathered}$$

the other four being obtained by using G_{24}. The fifteen conics each unaltered by a group of one system of fifteen octahedral groups in G_{360} are of the form

$$O \qquad \begin{gathered} x_1^2 + x_2^2 + x_2^3 \\ \frac{-\rho^2}{4}(x_1^2 + \varepsilon x_2^2 + \varepsilon^2 x_3^2) - \frac{1 + \bar{\rho}^2}{4} 2\,(x_2 x_3 + \varepsilon x_3 x_1 + \varepsilon^2 x_1 x_2, \\ \frac{2\bar{\rho}^2}{4} x_1^2 + \frac{2 + \rho^2}{4}(x_2^2 + x_3^2) + \frac{2 - \rho^2}{4} 2\, x_2 x_3 \end{gathered}$$

the others being obtained from the last two by the use of G_{24}.

Transforming these two sets by K and taking the point equations, which are the conjugates of the line equations, we have two new sets I' and O' corresponding to the second way of breaking up G_{360} into 6 and 15 ikosahedral and octahedral groups respectively. Those of I' are of the form

$$I' \qquad x_1^2 + x_2^2 - \bar{\rho}^2 x_3^2 + 2\,(\varepsilon - \varepsilon^2)\,x_1 x_2$$

and those of O' of the form

$$O' \qquad x_1^2 + x_2^2 + \frac{\bar{\rho}^2}{2} x_3^2$$

$$\frac{4+\bar{\rho}^2}{8}(x_1^2 + x_2^2) + \frac{-4+3\bar{\rho}^2}{8} x_3^2 + \frac{\bar{\rho}\,(2-\bar{\rho}^2)}{8}\, 2\,(x_1x_3 + x_2x_3) + \frac{\bar{\rho}^2}{8} x_1 x_2 .$$

The 42 *conics I, I', O, O', are the only conics which are their own second polars as to f^6 and as to ϕ^6.*

In the set I are twenty sets of three conics in involution corresponding to the twenty choices of three out of six. In the set I' are also twenty sets of three conics in involution. In O the first conic Σx_1^2 is in involution with the eight of the second form. These eight divide into four sets of two with each of which Σx_1^2 forms a set of three in involution. Hence Σx_1^2 occurs in 4 sets of three and there are then in O $15 \,.\, 4$ such sets, but each counted three times. Hence

In O, O', I, I', there are 80 *sets of three conics in involution, twenty sets in each.*

Now in any linear system of three conics there are just four such sets of three conics in involution. They are the polars of the vertices of the four inflexional triangles of the cubic whose conic polar system is the given system of conics. If the cubic is harmonic two sets of three are conjugate under the G_{36} of the cubic, and the other two sets of three are also conjugate. There are in G_{360} ten G_{36} which define ten harmonic cubics which in turn define 40 sets of three conics in involution. Twenty of these are conjugate and are found in I, the other twenty are also conjugate and are found in G. We have besides the ten Hessians of the harmonic cubic which are also harmonic and define the twenty sets in I' and the twenty sets in O'.

§5. *The Forms f^{2n} and ϕ^{2n} in Higher Domain.*

Without making any pretense to generality of treatment we find, however, in the higher domains examples of covariant quantic-pairs which are essentially different from those already noted. The cases in the binary and ternary domain are all alike in that they have no absolute invariant and that the one quantic is transformable into the other by a correlation. It is easy to obtain in the quaternary domain a case in which the first property does not exist and in the quinary domain a case in which the second does not exist. Thus the pair of quartic surfaces

$$\begin{aligned} f^4 &\equiv x_1^4 + x_2^4 + x_3^4 + x_4^4 + 12h\, x_1x_2x_3x_4 \\ \phi^4 &\equiv u_1^4 + u_2^4 + u_3^4 + u_4^4 + \frac{12}{h}\, u_1u_2u_3u_4 \end{aligned} \tag{1}$$

are a covariant quantic-pair with the irrational absolute invariant h.

In the quinary domain we assume a simplified pair of quartics unaltered by the cyclical advance of the variables and their changes of sign. Take

$$\begin{aligned} f^4 \equiv a\,(x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4) &+ 6b\,(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2x_5^2 + x_5^2x_1^2) \\ &+ 6c\,(x_1^2x_3^2 + x_2^2x_3^2 + x_4^2x_5^2 + x_4^2x_1^2 + x_5^2x_2^2) \\ \phi^4 \equiv \alpha\Sigma u_1^4 + 6\beta\Sigma u_1^2u_2^2 &+ 6\gamma\Sigma u_1^2u_3^2. \end{aligned} \tag{2}$$

Every quadric will have for polar as to f^4 a quadric whose polar as to ϕ^4 is the original quadric provided

$$\begin{aligned} a\alpha + 2b\beta + 2c\gamma &= 4b\beta = 4c\gamma \neq 0 \\ a\beta + b\alpha + c\beta + b\gamma + c\gamma &= 0 \\ a\gamma + c\alpha + c\beta + b\gamma + b\beta &= 0 \end{aligned} \tag{3}$$

From the first equation we have $a\alpha = 0$.

Suppose $\alpha = 0 \quad a \neq 0$. The solution is

$$a = 1 \quad b = c = -1/3 \quad \alpha = 0 \quad \beta = \gamma$$

and we have the covariant quantic-pair

$$\begin{aligned} f^4 &\equiv \Sigma x_1^4 - 2\Sigma x_1^2\,(x_2^2 + x_3^2) \\ \phi^4 &\equiv \qquad\quad \Sigma u_1^2\,(u_2^2 + u_3^2) \end{aligned} \tag{4}$$

f^4 has 30 double points of the type 0, 0, 0, 1, ± 1 while ϕ^4 has only five double flats of the type 0, 0, 0, 0, 1.

We have then the curious example of two quantics each the same covariant of the other and yet not transformable into each other.

If we solve the system of equations (3) more symmetrically by assuming both $a = 0$ and $\alpha = 0$ we have the quantic-pair

$$\begin{aligned} f^4 &\equiv \Sigma x_1^2 x_2^2 + \rho \Sigma x_1^2 x_3^2 \\ \phi^4 &\equiv \Sigma u_1^2 u_2^2 + \rho^2 \Sigma u_1^2 u_3^2. \end{aligned} \qquad \rho = \frac{-1 \pm i\sqrt{3}}{2} \tag{5}$$

We shall close with a discussion, in the next paragraph, of the covariant quantic-pair (5) which has properties entirely analogous to the pairs found in the binary and ternary domain.

§6. *The Finite Collineation Group, G_{25920}, in Five Variables.*[*]

We see that the quantics f^4 and ϕ^4 in (5) §5 are unaltered by the collineations

$$S \equiv \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = x_5 \\ x_5' = x_1 \end{cases} \qquad T \equiv \begin{cases} x_1' = x_1 \\ x_2' = \rho^2 x_3 \\ x_3' = \rho x_2 \\ x_4' = \rho x_5 \\ x_5' = \rho^2 x_4 \end{cases} \qquad U \equiv \begin{cases} x_1' = x_1 \\ x_2' = x_5 \\ x_3' = x_4 \\ x_4' = x_3 \\ x_5' = x_2 \end{cases}$$

S, T, and U generate an ikosahedral group G_{60}, which permutes evenly the five points and five flats of the pentahedron of reference, P_0. This G_{60} combined with the G_{16} made up of the changes of sign of the variables gives rise to a G_{960} which leaves P_0 unaltered. There are no other collineations which do not alter f^4 and P_0.

The vertices of P_0 are double points, *DP's* of f^4 and the flats of P_0 are double flats, *DE's*, of ϕ^4. We find without difficulty that f^4 has just 45 *DP*, 9 of the form

$$1,\ 0,\ 0,\ 0,\ 0 \qquad 0,\ \pm 1,\ \pm \rho,\ \pm \rho,\ \pm 1 \tag{1}$$

the 36 others being obtained from these by the use of S.

Also ϕ^4 has 45 *DE* whose coordinates are the conjugates of the coordinates of the 45 *DP*. The *DP* are all of the same type—the polar of each as to f^4 breaks up into a flat, the conjugate *DE*, and a hyper-cone with double point at the *DP* in question. The dual statement holds for the 45 *DE*. We have then a correspondence between the *DP* and *DE*, each corresponding to its conjugate. There is no trouble in verifying from (1) that

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[*] A complete enumeration of the subgroups of this group is given by L. Dickson, Transactions American Mathematical Society, Vol. 5, 1904, p. 126. Further references are given there, the articles by Burkhardt being especially pertinent to this account.

Each DE passes through 12 *of the* 45 *DP and each DP lies in* 12 *of the* 45 *DE*. Every collineation that leaves a *DE* unaltered must leave the corresponding *DP* unaltered and hence the twelve *DE* through it. In the *DE*, $x_1 = 0$, we have twelve points dividing into 3 tetrahedra namely.*

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{cccc} 1 & \rho & \rho & 1 \\ 1 & \rho & -\rho & -1 \\ 1 & -\rho & \rho & -1 \\ 1 & -\rho & -\rho & 1 \end{array} \qquad \begin{array}{cccc} 1 & \rho & \rho & -1 \\ 1 & \rho & -\rho & 1 \\ 1 & -\rho & \rho & 1 \\ -1 & \rho & \rho & 1 \end{array} \qquad (2)$$

And the twelve *DE* through $u_1 = 0$ cut out on $\chi_1 = 0$ the twelve planes (2′) of the above three tetrahedra, whose coordinates are the conjugates of the coordinates of the points (2). The three tetrahedra (2) form the well known desmic system of tetrahedra, which is unaltered by a group of 576 collineations. If we require that f^4 also be unaltered by the collineations then the group can permute the variables only in the even ways and its order is reduced to 288. However, a change of sign of x_1 being added to the group the order is restored to 576. This G_{576} is generated by the tetrahedral subgroup of G_{60} which leaves $x_1 = 0$ unaltered, by the G_{16} of the changes of sign of the variables and by the collineation S_a of period 3, whose square is S_α

$$S_a \equiv 2x' = \begin{array}{c} \\ \\ \rho \\ \rho \\ \\ \end{array}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & \rho^2 & \rho^2 & 1 \\ 0 & 1 & \rho^2 & -\rho^2 & -1 \\ 0 & 1 & -\rho^2 & \rho^2 & -1 \\ 0 & -1 & \rho^2 & \rho^2 & -1 \end{pmatrix} x \qquad S_\alpha \equiv 2x' = \begin{array}{c} \\ \\ \rho \\ \rho \\ - \end{array}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & \rho^2 & \rho^2 & -1 \\ 0 & 1 & \rho^2 & -\rho^2 & 1 \\ 0 & 1 & -\rho^2 & \rho^2 & 1 \\ 0 & -1 & \rho^2 & \rho^2 & 1 \end{pmatrix}$$

The factors outside the matrix are supposed to multiply each element. In the row opposite, S_a and S_α permute the three tetrahedra (2) cyclically. They are of determinant unity and in the usual canonical form. They leave the configuration of 45 *DP* unaltered and therefore f^4 as well. Hence

(3) *Of the group of collineations which leaves* f^4 *unaltered there is a* G_{576} *which leaves one of the* 45 *DP unaltered.* f^4 *then is automorphic under a group of order* 45. 576 = 25920.

Since the pentahedron of reference P_0 is unaltered by a G_{960} we should expect it to be transformed by G_{25920} into $\frac{25920}{960}$ or 27 others. The rows of the matrices of S_a and S_α determine two others, P_a and P_α. Both P_a and P_α contain the *DE*, $x_1 = 0$, as well as P_0. If each *DE* is contained in 3 pentahedra we should have again $\frac{45\,.\,3}{5} = 27$ pentahedra. These pentahedra are obtained from the matrices of S_a, S_α and S_0 where

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* For a complete geometrical discussion of this system and the G_{288} see J. Feder, Math. Annalen, Vol. 47, p. 375.

$$S_0 \equiv x' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} x$$

and the matrices of twenty-four other collineations derived as follows :— Transform the collineations S_a and S_α by the collineation S of G_{60}. We have

$$S_b \equiv 2x' = \begin{array}{r} \\ \\ \\ \rho \\ \rho \end{array}\begin{pmatrix} -1 & 0 & -1 & \rho^2 & \rho^2 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & \rho^2 & \rho^2 \\ -1 & 0 & 1 & \rho^2 & -\rho^2 \\ -1 & 0 & 1 & -\rho^2 & \rho^2 \end{pmatrix} x \qquad S_\beta \equiv 2x' = \begin{array}{r} - \\ \\ \\ \rho \\ \rho \end{array}\begin{pmatrix} 1 & 0 & -1 & \rho^2 & \rho^2 \\ 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 1 & \rho^2 & \rho^2 \\ 1 & 0 & 1 & \rho^2 & -\rho^2 \\ 1 & 0 & 1 & -\rho^2 & \rho^2 \end{pmatrix} x$$

$$S_c \equiv 2x' = \begin{array}{r} \rho \\ \\ \\ \\ \rho \end{array}\begin{pmatrix} \rho^2 & -1 & 0 & 1 & -\rho^2 \\ \rho^2 & -1 & 0 & -1 & \rho^2 \\ 0 & 0 & 2 & 0 & 0 \\ \rho^2 & 1 & 0 & 1 & \rho^2 \\ -\rho^2 & -1 & 0 & 1 & \rho^2 \end{pmatrix} x \qquad S_\gamma \equiv 2x' = \begin{array}{r} \rho \\ - \\ \\ \\ \rho \end{array}\begin{pmatrix} \rho^2 & 1 & 0 & 1 & -\rho^2 \\ \rho^2 & 1 & 0 & -1 & \rho^2 \\ 0 & 0 & 2 & 0 & 0 \\ \rho^2 & -1 & 0 & 1 & \rho^2 \\ -\rho^2 & 1 & 0 & 1 & \rho^2 \end{pmatrix} x$$

$$S_d \equiv 2x' = \begin{array}{r} \rho \\ \rho \\ \\ \\ \\ \end{array}\begin{pmatrix} \rho^2 & -\rho^2 & -1 & 0 & 1 \\ -\rho^2 & \rho^2 & -1 & 0 & 1 \\ \rho^2 & \rho^2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ \rho^2 & \rho^2 & 1 & 0 & 1 \end{pmatrix} x \qquad S_\delta \equiv 2x' = \begin{array}{r} \rho \\ \rho \\ - \\ \\ \\ \end{array}\begin{pmatrix} \rho^2 & -\rho^2 & 1 & 0 & 1 \\ -\rho^2 & \rho^2 & 1 & 0 & 1 \\ \rho^2 & \rho^2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ \rho^2 & \rho^2 & -1 & 0 & 1 \end{pmatrix} x$$

$$S_e \equiv 2x' = \begin{array}{r} \\ \rho \\ \rho \\ \\ \\ \end{array}\begin{pmatrix} 1 & \rho^2 & \rho^2 & 1 & 0 \\ 1 & \rho^2 & -\rho^2 & -1 & 0 \\ 1 & -\rho^2 & \rho^2 & -1 & 0 \\ -1 & \rho^2 & \rho^2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} x \qquad S_\epsilon \equiv 2x' = \begin{pmatrix} 1 & \rho^2 & \rho^2 & -1 & 0 \\ 1 & \rho^2 & -\rho^2 & 1 & 0 \\ 1 & -\rho^2 & \rho^2 & 1 & 0 \\ -1 & \rho^2 & \rho^2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} x$$

In S_b change the signs of the first and third columns of the matrix and call the collineation S'_b. With the collineation S^2TS^2 of G_{60} form the combination $S^2TS^2S'_bS_\alpha$. The result is

$$S_1 \equiv 2\rho x' = \begin{pmatrix} 0 & 1 & \rho^2 & \rho^2 & 1 \\ \rho^2 & \rho^2 & 1 & 0 & 1 \\ 1 & 0 & 1 & \rho^2 & \rho^2 \\ 1 & \rho^2 & \rho^2 & 1 & 0 \\ \rho^2 & 1 & 0 & 1 & \rho^2 \end{pmatrix}$$

The matrix of this is of different type from the eleven already found and we find by changing the signs of the columns in all possible ways 15 new pentahedra which with that of S_1 we shall name $P_1, P_2, \ldots., P_{16}$ and the corresponding collineations $S_1, \ldots. S_{16}$. Then $S_1 \,.\, G_{16} = S_1, S_2, \ldots. S_{16}$. These results may be incorporated in the theorem ;

(4) *There are* 27 *pentahedra whose vertices belong to the* 45 *DP and whose flats belong to the DE. Each of the five flats contains besides the four vertices eight DP, each of the five points lies on eight DE in addition to the four flats of the pentahedron, i. e., the five flats contain the* 45 *DP and the five vertices lie on the* 45 *DE. The* G_{25920} *contains one system of* 27 *conjugate subgroups of order* 960 *each subgroup having an invariant pentahedron.*

The fact that f^4 and ϕ^4 have different forms when referred to the same pentahedron, P_0, does not lead to the conclusion, as one might suppose, that there are two systems of 27 conjugate G_{960}. For the correlation, K,

$$x_1 = u_1, \quad x_2 = \rho^2 u_5, \quad x_3 = \rho u_3, \quad x_4 = \rho u_4, \quad x_5 = \rho^2 u_2$$

transforms f^4 into ϕ^4 without altering P_0. It differs from the collineations that do not alter P_0 in that the vertices of P_0 are carried into an *odd* permutation of the flats of P_0.

We see at once that the G_{60} generated by S, T, and U, since it contains no changes of sign of the variables, must also effect the even permutations of the flats of P_1 . P_0 and P_1 have no common vertices. The same is true of P_0 and P_2, P_0 and P_3, . . . , P_0 and P_{16}. But since P_1 is transformed into P ($\iota = 1, \ldots 16$) by G_{16} we find for each pair P_0 and P_ι a different G_{60} the transform of the original G_{60} by some collineation of G_{16}. Hence

(5) G_{25820} *contains a system of* 1/2 27.16 *conjugate* G_{60}, *any member of the system being defined as the group which leaves each of two pentahedra, having no common vertex, unaltered.*

The collineation S_1 is of period 4. Its square is T of the original G_{60}. Hence S_1 permutes P_0 and P_1 and with G_{60} generates a G_{120} isomorphic with the symmetric group of five things. One such set of five things is P_a, P_b, P_c, P_d, P_e which are the only pentahedra each containing a flat of P_0 and a flat of P_1. If we call P_2, P_4, P_6, P_8, P_{10} the pentahedra which arise from P_1 by changing the sign of the first, second, . . . , fifth columns respectively we see that P_2, P_4, P_6, P_8, P_{10} each contains one of the flats of P_1 and that P_α, P_β, P_γ, P_δ, P_ϵ each contains one of the flats of P_0. Hence the first set of five pentahedra is permuted evenly by G_{60} and the same is true of the second set while the odd substitutions of G_{120} permute the two sets.

(6) *Hence in* G_{25920} *there is a system of* 27.8 *conjugate* G_{120} *each of which is defined as the group which does not alter a set of two pentahedra having no common vertex.*

We consider now the following group of twelve pentahedra which divide into two sets of six, each in one set being paired with a "partner" in the other,

$$D_6 \equiv \begin{Bmatrix} P_0, & P_2, & P_4, & P_6, & P_4, & P_{10} \\ P_1, & P_\alpha, & P_\beta, & P_\gamma, & P_\delta, & P_\epsilon \end{Bmatrix}$$

Each P in the upper line has a flat and vertex in common with every P in the lower line except its partner. The choice of any pair determines the five others. The pair P_0, P_1 being unaltered by a G_{120} which simply permutes the other five pairs we can conclude that the entire configuration D_6 is unaltered by a G_{720}. We can easily construct a unique quadric which admits the original G_{60} namely

$$Q \equiv \rho^2 (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) + \rho (x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2) \qquad (7)$$

But this passes through the vertices not only of P_0 but also of all the other pentahedra of D_6. The equation in flats is the conjugate of (7) whence it touches all the flats of D_6. (7) must be unaltered therefore by the entire G_{720} and be transformed by G_{25920} into only 36 quadrics. 16 of the 36 are obtained from (7) by using G_{16}. Forming (7) for the pentahedron P_α we have

$$Q \equiv \rho x_3^2 + x_4^2 + \rho x_5^2 \pm 2\rho^2 x_1x_2 \qquad (8)$$

and applying T of G_{60} to this we have

$$Q \equiv x_2^2 + \rho^2 (x_4^2 + x_5^2) \pm 2\rho x_1x_3 \qquad (9)$$

From the four quadrics (8) and (9) we obtain by the use of S the twenty remaining quadrics Q.

(10) *There are then thirty-six quadrics Q, each of which passes through the thirty vertices and is touched by the thirty flats of a D_6. The flat and vertex forming an element of Q occurs in two pairs of D_6. There are in G_{25920} 36 conjugate G_{720} one corresponding to each Q and its inscribed D_6.*

Another noteworthy configuration associated with f^4 and ϕ^4 is a system of points N and flats F. One of the flats F has the equation

$$x_1 + \rho^2 x_2 + \rho^2 x_5 = 0 \qquad (11)$$

and is defined by the property it possesses of passing through 18 of the 45 DP.

To find the number of the F we observe that through one DP, *e.g.*, 1, 0, 0, 0, 0 pass at least 16 F, namely the four

$$\begin{array}{ccccc} 0 & \rho^2 & 1 & \rho^2 & 0 \\ 0 & 0 & \rho^2 & 1 & \rho^2 \\ 0 & \rho & \rho & \rho & 1 \\ 0 & 1 & 0 & \rho & \rho \end{array}$$

and the twelve obtained from these by changing the signs of their coordinates. Each of these sixteen passes through a "line of the desmic system of tetrahedra" in $x_1 = 0$, *i. e.*, a line of the system containing one vertex of each of the three tetrahedra. There are just 16 such lines, one in each of the 16 F. The line and DP do not fix the F, but it can easily be verified that through the DP and the line there can be only one F that contains 18 DP. Hence the number of the flats F is $\frac{16.45}{18} = 40$. They may all be obtained from the sixteen mentioned by the use of S.

(12) *The* 45 *DP lie by* 18*'s on* 40 *flats F; the* 45 *DE pass by* 18*'s through* 40 *points N whose coordinates are conjugate to those of F. No N lies on an F; no F passes through an N. In* G_{25920} *is a system of* 40 *conjugate* $G_{3.216}$, *each having for invariant an F and an N.*

The G_{25920} has been given [*] as a group in four variables containing as subgroups the group G_{216} in three variables. In our case the subgroup which leaves an F invariant appears in that three dimensional space as a collineation group projectively different from the ternary G_{216}. The subgroup in F is indeed the group $G_{3.216}$ of the three dimensional tetrahedral cubic $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$. We verify this as follows:—Assuming a flat F as invariant the remaining 39 break up into a set of 12 and a set of 27. The set of 12 breaks up into 4 sets of three, each one of a set of three cutting F in the same plane and the 12 therefore giving rise to a tetrahedron on F. Hence

(13) *The* 18 *DP on F lie by* 9*'s on the four planes of a tetrahedron in F; two planes pass through each DP, hence on each edge of the tetrahedron lie three of the DP. The dual of this is true for an N.*

We also have that

(14) *The* 40 *F meet by* 4*'s in forty planes H. The* 40 *N lie by* 4*'s on forty lines L.*

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[*] Maschke, Math. Annalen, p. 317. Further references are given there.

The remaining 27 F above mentioned are thus accounted for: On each of three edges of the tetrahedron in F that meet in a point are three DP which determine 27 planes, the planes of intersection of the 27 F with the invariant F. This is then exactly the configuration determined by the tetrahedral cubic surface.

For later use we may note also that there is through every pair of points DP which do not form vertices of the same P a line containing a third DP contained in no pentahedra containing the first two DP. We may call such a set of three a "line-triad." As an example we have that the three points

$$\begin{array}{rll} 1,\ 0,\ 0,\ 0,\ 0 & \text{contained in} & P_a,\ P_o,\ P_\alpha \\ 1,\ 0,\ 1,\ \rho,\ \rho & \text{“} \qquad \text{“} & P_1,\ P_b,\ P_4 \\ -1,\ 0,\ 1,\ \rho,\ \rho & \text{“} \qquad \text{“} & P_2,\ P_\beta,\ P_3 \end{array} \tag{15}$$

lie on a line. The 9 pentahedra are determined however by a second "line-triad" for

$$\begin{array}{rrrrrll} 0, & 1, & 0, & 0, & 0 & \text{is contained in} & P_0,\ P_b,\ P_\beta \\ 0, & 1, & \rho, & \rho, & 1 & \text{“} \qquad \text{“} \qquad \text{“} & P_a,\ P_1,\ P_2 \\ 0, & -1, & \rho, & \rho, & 1 & \text{“} \qquad \text{“} \qquad \text{“} & P_a,\ P_4,\ P_3 \end{array} \tag{16}$$

We shall say that one line-triad determines uniquely its "conjugate line-triad." The two lines are skew and determine therefore a flat, one of the forty F, $\rho^2 x_3 + x_4 + \rho^2 x_5 = 0$. In F the two lines are opposite edges of the tetrahedron [see (13)] and hence a pair of conjugate line-triads determine uniquely two other pairs which are reciprocally related to it. We find in all 240 triads since there are forty F. But it is clear also that through a DP pass 16 line-triads whence their number is $\frac{45.16}{3} = 240$.

(17) *There are then* 40 *sets of reciprocally related three pairs of conjugate line-triads.*

We have just seen that one line-triad, l, determines a conjugate and the two a flat F. In any flat F are 42 l's, the six edges of the tetrahedron of (13) and 9 in each of the faces. Hence through each l pass $\frac{42.40}{240} = 7$ F, a number that is easily recognized for each of the 42 lines in an F. Thus through the l of (15) pass the flats F

$$\begin{array}{ccccc} 0 & 0 & \rho^2 & 1 & \rho^2 \\ \hline 0 & \rho^2 & -1 & \rho^2 & 0 \\ 0 & \rho & \rho & 0 & -1 \\ 0 & 1 & 0 & -\rho & \rho \\ \hline 0 & \rho^2 & 1 & -\rho^2 & 0 \\ 0 & \rho & -\rho & 0 & 1 \\ 0 & 1 & 0 & \rho & -\rho \\ \hline \end{array} \tag{18}$$

We note that the first is singled out by lying in a pencil with each of the two sets of three. This first is exactly the F determined by the pair of conjugate l, (15) and (16). Hence

(19) *In each F are 42 l; through each l are 7 F which form two pencils of 4, the two pencils having a common F. Six l determine the same F, the six forming the edges of a tetrahedron in F. The forty tetrahedra arising from the forty F contain as edges all the 240 lines.*

Besides the 40 F we have in (14) a set of 40 conjugate planes H, the faces of the 40 tetrahedra in F, each face occuring in 4 F that form a pencil. Each of the 40 H are unaltered by a $G_{3.216}$ and we wish to find whether it is contained in the set of conjugate $G_{3.216}$ already found. But the $G_{3.216}$ of (12) has for invariant a tetrahedron in the invariant F and it is not difficult to show that the remaining 39 F under the subgroup which leaves a face of the tetrahedron unaltered break up into a set of three, a set of 9, and a set of 27 F, the members of any set being conjugate. None of the 39 are unaltered whence the group that permutes four F that form a pencil (*i. e.*, that has for invariant an H) has no invariant F and is distinct from the $G_{3.216}$ of (12). Since H is common to 4 F and in each F is a tetrahedron peculiar to itself it follows that the 12 l in a face of the tetrahedron of an F lie in 4 triangles, each triangle belonging to the tetrahedron of one of the 4 F. Hence in H we have the configuration of triangles associated with a syzygetic pencil of cubics.

(20) *The G_{25920} contains two different systems of forty $G_{3.216}$. One is the system described in (12); the other is the system which has for invariant configurations the forty planes H or the forty lines L. In an invariant plane H, the invariant configuration is the same as that of the ternary G_{216}.*

We have obtained above the subgroups of G_{25920} of smaller indices. It is not intended to attempt here a complete classification of the various subgroups of G_{25920}. But it is clear that for such a purpose the above representation of the group is well suited. The variety of geometrical figures that belong to certain subgroups and that are of simple projective character is very great. For example, besides the flats F through 18 DP and the flats DE through 12 DP, there are flats of the type $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ through 10 DP, flats of the type $x_2 + x_5 = 0$ through 7 DP, and flats of the type $x_2 + x_3 + x_4 = 0$ through 6 DP. Also the 40 points N, dual to the F have barely been mentioned.

That G_{25920}, abstractly considered, is isomorphic with the group of the equations of the 27 lines of a cubic surface has long been known.* A few simple correspondence principles permit of deriving the more important properties of these 27 lines from the configuration described above. These are :—

(*a*) *To a line* λ *of the cubic surface corresponds a pentahedron* P.

(*b*) *To two lines* λ *which intersect* (*lie in a plane* ε) *correspond two pentahedra* P *with a common vertex, a* DP.

With the aid of (*a*) and (*b*) the collection of facts concerning the 27 λ given in Pascal's Repertorium† can be obtained without difficulty from the properties of the 27 P. Thus from the fact that each P has 5 vertices and each vertex belongs to 3 P we see that

1° *There are* 45 ε *each containing* 3λ. *Each* λ *occurs in* 5 ε.

P_0 has a vertex in common with $P_a, \ldots, P_b$ and $P_\alpha, \ldots, P_\epsilon$ but not with $P_1, \ldots, P_{16}$. Hence

2° *Each* λ *cuts* 10 *other* λ *and does not cut the remaining* 16. *Two* λ *that do not cut will both be cut by* 5λ, *each be cut by* 5 *other* λ, *and neither be cut by the remaining* 10λ.

The latter part of this is evident from the fact that P_0 and P_1 have a vertex in common with $P_a, \ldots, P_e$, that P_0 has one in common with $P_\alpha, \ldots, P_\epsilon$, and that P_1 has one in common with $P_2, P_4, P_6, P_8, P_{10}$. From (6) we have

3° *There are* 216 *skew-pairs* λ.

The existence of the D_6 (p. 50) shows that

4° *There are* 36 *"double-sixes."*

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* Jordan, Traité Des Substitutions, p. 316.

† *E.* Pascal, Repertorium Der Höheren Mathematik, II, p. 284.

From the double sixes we can pass at once to the various systems of non-cutting lines mentioned by Pascal. That the quadric Q (7) has the same form as fifteen others but a different form from that of the twenty (8) and (9) leads to the theorem

5° *A double six has* 4λ *in common with each of* 15 *others and* 6λ *in common with each of* 20 *others.*

Two DP not contained in a P lie on a line with a third DP; this line determines another such that the mine P having vertices at the 3 DP are the same for the two sets of three. There are 120 such line pairs. Each pair determines an F and in F is one of the three pairs of opposite edges of a tetrahedron. The 18 points on the six lines are the vertices of 54 P, each of the 27 P occurring twice (see (15), (16), (17), (18), (19)). The translation of the above results in

6° *Two* ε *having no* λ *common to each determine a third* ε, *no two of the three having a* λ *common to each;* 9λ *in the* 3ε *are contained similarly in a second set of three* ε. *The two sets of* 3ε *are called a "pair of conjugate trihedra." There are* 120 *such pairs. Any one pair of conjugate trihedra determines two other pairs such that the complex of three pairs contains the* 27λ. *There are* 40 *such complexes.*

Finally, the considerations which led to the second svstem of 40 conjugate $G_{3.216}$ may be translated as follows:—

(21) *A complex of conjugate trihedra-pairs contains* 18ε. *These* 18ε *may be divided in four ways into two sets of* 9ε, *a set* A *and a set* B *such that the set* A *contains the* 27λ *and the set* B *contains the* 27λ. *There are* 160 *sets* A *and* 160 *sets* B. *The* 160 *sets* A *are each common to a group of* 4 *complexes. There are* 40 *"complex-groups of four,"* i. e., 4 *complexes having a set* A *of* 9ε *common. However, no two complexes have a set* B *of* 9ε *common.*

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